Documents de Travail du
Centre d'Economie de la Sorbonne

CES Working Papers

Ensuring the boundedness of the core of games with restricted cooperation

Michel GRABISCH

2010.93

Maison des Sciences Économiques, 106-112 boulevard de L'Hôpital, 75647 Paris Cedex 13
http://centredeconomiesorbonne.univ-paris1.fr/bandeau-haut/documents-de-travail/
ISSN : 1955-611X

Ensuring the boundedness of the core of games with restricted cooperation

Michel GRABISCH
Paris School of Economics, University of Paris I
106-112, Bd de l'Hôpital, 75013 Paris, France
Tel (+33) 1-44-07-82-85, Fax (+33) 1-44-07-83-01
email michel.grabisch@univ-paris1.fr

Abstract

The core of a cooperative game on a set of players N is one of the most popular concept of solution. When cooperation is restricted (feasible coalitions form a subcollection $\mathcal{F}$ of 2^N), the core may become unbounded, which makes it usage questionable in practice. Our proposal is to make the core bounded by turning some of the inequalities defining the core into equalities (additional efficiency constraints). We address the following mathematical problem: can we find a minimal set of inequalities in the core such that, if turned into equalities, the core becomes bounded? The new core obtained is called the restricted core. We completely solve the question when $\mathcal{F}$ is a distributive lattice, introducing also the notion of restricted Weber set. We show that the case of regular set systems amounts more or less to the case of distributive lattices. We also study the case of weakly union-closed systems and give some results for the general case.

Keywords: cooperative game, core, restricted cooperation, bounded core, Weber set

1 Introduction

In cooperative models, one of the main issues is to define in a rational way the sharing of the total worth of a game among the players, what is usually called the *solution* of the game. The core is perhaps the most popular concept of solution, because it is built on a very simple rationality criterion: no coalition should receive less than that it can earn by itself, thus avoiding any instability in the game (this is often called *coalitional rationality*). The core is a bounded convex polyhedron whenever nonempty, and its properties have been studied in depth (see, e.g., [27, 22, 25]).

The classical setting of cooperative games stipulates that any player can (fully) participate or not participate to the game, and that any coalition can form. This too simplistic framework has been made more flexible in many respects, or more tailored to some special kind of application by many authors: let us cite on the one hand multichoice games [21, 26], games with multiple alternatives [3, 7] and bicooperative games [4, 23] (participation is gradual, can be positive or negative, or the player has several options), and

Documents de Travail du Centre d'Economie de la Sorbonne - 2010.93

on the other hand, games with restricted cooperation, where only a limited set of coalitions are allowed to form. A vast literature is devoted to this last category, studying various possibilities for the algebraic structure of the set of *feasible* coalitions: games on antimatroids [2], convex geometries [5], lattices [10, 12, 16], graphs [28, 30], etc.

Our study will concern games with restricted cooperation, and especially the core of such games. Here also, there is a vast literature we will not cite here (see a recent survey by the author on this topic [15]). Indeed, the study of the core in such a general situation becomes much more challenging: since the core is defined by a system of linear inequalities, it is always a polyhedron, however it need not be bounded any more, and it may even have no vertice or it may contain a line. As a matter of fact, since the core is supposed to represent a set of payoffs for players, boundedness is perhaps the property one wants to keep in priority (arbitrarily large payoffs cannot exist in reality). Therefore, a natural question arises: *How to make to core bounded in any case, keeping the spirit of its definition?* By "spirit of definition", we mean the essential idea of coalitional rationality. A very simple answer to this question was proposed by Grabisch and Xie [17, 18]: turn some of the inequalities into equalities, which can be seen as adding supplementary *efficiency* constraints, while preserving coalitional rationality. The authors proposed a systematic way of doing this for games on distributive lattices, according to some interpretation related to the hierarchy of players.

We want to take here a more general and mathematical point of view. Specifically, we address the following question: *Suppose v is a game with restricted cooperation, whatever the structure of its set of feasible coalitions. Can we find a minimal set of inequalities in the core of v such that, if turned into equalities, the core will be bounded?* A second question is: what about the Weber set? Can we define it so that the classical property of inclusion of the core into the Weber set is preserved?

We give a complete answer to these questions for games on distributive lattices, —thus generalizing and simplifying results of Grabisch and Xie, and partial answers for other structures and the general case.

The paper is organized as follows. Section 2 introduces the basic material for the paper: set systems, posets and lattices, etc. We also explain our main idea to make the core bounded. Section 3 studies the case of distributive lattices. It gives an optimal algorithm to find which inequalities must be turned into equalities. Also, it introduces the notion of restricted Weber set, and shows that the classical result of inclusion of the core into the Weber set still holds. Section 4 studies the general case. A first result shows that if rays have a certain form, one can treat an equivalent problem where the set system is a distributive lattice, and therefore benefit from results of Section 3. It is shown that regular set systems fall into this category. However, for weakly union-closed systems, an additional condition on the set system is required. We give also an algorithm to find all extremal rays of the core of a game on a regular set system.

We assume some familiarity of the reader with polyhedra. To avoid a heavy notation, we often omit braces and commas for singletons and sets, writing e.g, $N \setminus i$ instead of $N \setminus \{i\}$, 123 instead of $\{1, 2, 3\}$, etc.

2 Preliminaries

2.1 Games on set systems

We consider $N := \{1, \dots, n\}$ the set of players, agents, etc. A *set system* $\mathcal{F}$ on N is a collection of subsets of N containing N and $\emptyset$. One can think of $\mathcal{F}$ as the collection of *feasible coalitions*, and when $\mathcal{F} \subset 2^N$ it is common to speak of *restricted cooperation*. A *game* on $\mathcal{F}$ is a function $v : \mathcal{F} \to \mathbb{R}$ such that $v(\emptyset) = 0$.

A *payoff vector* x is any vector in $\mathbb{R}^N$, which defines the amount of money given to each player. It is common to use the notation $x(S)$ where $S \in 2^N$, as a shorthand for $\sum_{i \in S} x_i$, with the convention $x(\emptyset) := 0$. The *core* of a game v on $\mathcal{F}$ is the set of payoff vectors being *coalitionally rational*, in the sense that any feasible coalition S gets at least what it can achieve by itself, namely $v(S)$:

$$\mathcal{C}(v) := \{x \in \mathbb{R}^N \mid x(S) \geq v(S), \forall S \in \mathcal{F}, x(N) = v(N)\}.$$

The equality $x(N) = v(N)$ is known as the *efficiency condition*. It means that no more than $v(N)$ can be distributed among the players whatsoever, and distributing strictly less would be inefficient (the definition makes sense only if the grand coalition N is the best way to make profit).

By definition, the core is a closed convex polyhedron, however it may be unbounded (see in [15] a survey of the properties of the core of games on set systems). We denote by $\mathcal{C}(0)$ the *recession cone* of $\mathcal{C}(v)$, that is, the cone defined by

$$\mathcal{C}(0) := \{x \in \mathbb{R}^N \mid x(S) \geq 0, \forall S \in \mathcal{F}, x(N) = 0\}.$$

It is well known from the theory of polyhedra that $\mathcal{C}(v)$ is bounded if and only if $\mathcal{C}(0) = \{0\}$, and that the extremal rays of $\mathcal{C}(0)$ are the extremal rays of $\mathcal{C}(v)$ for any game v. Since in this paper we are mainly interested into the boundedness issue and therefore in rays, we mainly deal with the recession cone $\mathcal{C}(0)$.

2.2 Posets and lattices

A set system $\mathcal{F}$ can be seen as a partially ordered set (poset) when endowed with the inclusion order. Properties of $\mathcal{C}(v)$ substantially differ according to the algebraic structure of $(\mathcal{F}, \subseteq)$. We give here some fundamental notions on posets which will be useful in the sequel (see, e.g., Davey and Priestley [8] for details).

A *partially ordered set* $(P, \leq)$, or *poset* for short, is a set P endowed with a partial order $\leq$ (reflexive, antisymmetric and transitive). As usual, the asymmetric part of $\leq$ is denoted by $<$. For $x, y \in (P, \leq)$ (if no ambiguity occurs, we may write simply P), we write $x \prec y$ and say that x is *covered* by y if $x < y$ and there is no $z \in P$ such that $x < z < y$. An element $x \in P$ is minimal if there is no $y \in P$ such that $y < x$.

A *chain* from x to y in P is any sequence $x, x_1, \dots, x_p, y$ of elements of P such that $x < x_1 < \cdots < x_p < y$. The chain is *maximal* if no other chain from x to y contains it, i.e., if $x \prec x_1 \prec \cdots \prec x_p \prec y$. The *length* of a chain is its number of elements minus 1.

The *height* of $x \in P$, denoted by $h(x)$, is the length of a longest chain from a minimal element to x. Elements of same height l form *level* $l + 1$. Hence, level 1 (denoted by L_1) is the set of all minimal elements, level 2 (denoted by L_2) is the set of minimal elements of $P \setminus L_1$, etc. The height of P, denoted by $h(P)$, is the maximum of $h(x)$ taken over all elements of P.

Consider a poset $(P, \leq)$ and some $Q \subseteq P$. Then Q is a *downset* of P if $x \in Q$ and $y \leq x$ imply $y \in Q$. Any element $x \in P$ generates a downset, defined by $\downarrow x := \{y \in P \mid y \leq x\}$. We denote by $\mathcal{O}(P)$ the set of all downsets of P.

A lattice $(L, \leq)$ is a poset having the following property: for any $x, y \in L$, their supremum and infimum, denoted by $x \vee y$ and $x \wedge y$, exist in L. When a lattice is finite, it has a greatest element $\top = \bigvee_{x \in L} x$ (top element), and a smallest element $\perp = \bigwedge_{x \in L} x$ (bottom element). If $\vee, \wedge$ obey distributivity, then L is said to be *distributive*. In a distributive lattice L, all maximal chains from $\perp$ to $\top$ have same length $h(L)$.

Given a lattice L, an element $x \in L$, $x \neq \perp$, is *join-irreducible* if it cannot be expressed as the supremum of other elements, or equivalently, if it covers a unique element. We denote by $\mathcal{J}(L)$ the set of join-irreducible elements of L. It can be shown that if L is distributive, its height $h(L)$ equals the number of join-irreducible elements $|\mathcal{J}(L)|$.

Finite distributive lattices have a remarkable property: they are completely determined by their join-irreducible elements. Specifically, consider $(L, \leq)$ a distributive lattice, and $(\mathcal{J}(L), \leq)$ its join-irreducible elements considered as a subposet of L. Then Birkhoff's theorem [6] says that $(L, \leq)$ and $(\mathcal{O}(\mathcal{J}(L)), \subseteq)$ are isomorphic. Conversely, any poset $(P, \leq)$ *generates* a distributive lattice $(\mathcal{O}(P), \subseteq)$ (hence, we deduce a characterization of distributive lattices: the set of distributive lattices of height n is in bijection with the set of posets of n elements). Figure 1 illustrates this fundamental result.

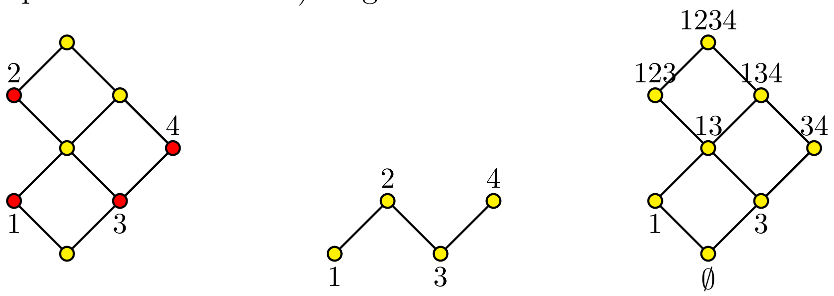


Figure 1: Left: a distributive lattice L. Join-irreducible elements are those in dark grey. Center: the poset $\mathcal{J}(L)$ of join-irreducible elements. Right: the set $\mathcal{O}(\mathcal{J}(L))$ of all downsets of $\mathcal{J}(L)$ ordered by inclusion, which is isomorphic to L

2.3 Main families of set systems

Among the numerous families of sets systems, we put emphasis on three of them: distributive lattices of height n, regular set systems, and weakly union-closed systems.

$(\mathcal{F}, \subseteq)$ is a distributive lattice is equivalent to say that $\mathcal{F}$ is a set system closed under union and intersection, and by Birkhoff's result, it is generated by a poset of n elements if and only if $\mathcal{F}$ has height n. This poset can be interpreted as the set of players N endowed with some partial order $\leq$, which can be thought of as a hierarchy on players or a precedence order (and then we recover exactly games with precedence constraints of Faigle and Kern [12]): see again Figure 1, center (the hierarchy) and right (the set system).

Remark 1. We discard from the analysis distributive lattices L of height smaller than n: essentially, it amounts to redefine the set of players as N' with $|N'| = h(L)$, where some of the players of N have been regrouped into "macro-players".

A set system is *regular* if all its maximal chains from $\emptyset$ to N are of length n (see [19, 20, 24] for works dealing with regular set systems). Evidently, any set system closed under union and intersection of height n is regular, but the converse is not true.

A set system $\mathcal{F}$ is *weakly union-closed* if for any $S_1, S_2 \in \mathcal{F}$ such that $S_1 \cap S_2 \neq \emptyset$, we have $S_1 \cup S_2 \in \mathcal{F}$ (see [11] and also [1] where weakly union-closed systems are called union stable structures).

These three families are distinct, and no family is included into another one (see [15]).

2.4 How to make the core bounded

Given a set system $\mathcal{F}$, our main goal is to modify the definition of the core to make it bounded for any game v, by replacing some of the inequalities by equalities (evidently, the core will become bounded after a sufficient number of such operations). Observe that doing so preserves the coalitional rationality principle, and this can be interpreted as adding new efficiency constraints.

We call *normal sets* the sets $S \in \mathcal{F}$ corresponding to inequalities $x(S) \geq v(S)$ turned into equalities $x(S) = v(S)$, provided the collection of those sets makes the core bounded (recall that this is independent of v since it suffices to study the recession cone $\mathcal{C}(0)$). Such a collection (called a *normal collection*), is denoted by $\mathcal{N} := \{N_1, \ldots, N_q\}$, and we make the convention that N is *not* an element of $\mathcal{N}$. We call the core with these additional equalities the *core restricted by the normal collection* $\mathcal{N}$, or if no ambiguity occurs, the *restricted core*, and denote it by $\mathcal{C}_{\mathcal{N}}(v)$.

As mentionned in the introduction, Grabisch and Xie have proposed a particular way to define a normal collection when $\mathcal{F}$ is a distributive lattice. Suppose $\mathcal{F}$ is a distributive lattice of height n, with generating poset $(N, \leq)$. As mentionned in Section 2.2, the height function on $(N, \leq)$ induces a partition of N into levels $L_1, \ldots, L_q$. Then the normal collection of Grabisch and Xie is simply $(L_1, L_1 \cup L_2, \ldots, L_1 \cup \cdots \cup L_{q-1})$. Note that the obtained normal collection is *nested*, i.e., it forms a chain in $\mathcal{F}$.

3 Case of distributive lattices of height n

3.1 Normal sets

We know from the previous section that these set systems are closed under union and intersection, that they possess n join-irreducible elements, and that they are generated by a poset $(N, \leq)$ (i.e., $\mathcal{F} = \mathcal{O}(N, \leq)$). We recall that $i \prec j$ means that $i < j$ and there is no $k \in N$ such that $i < k < j$.

For those sets systems, we know the following result from Tomizawa [29]. We denote by J_i, $i \in N$, the join-irreducible element of $\mathcal{F}$ induced by i, that is simply, $J_i = \downarrow i$.

Theorem 1. The extremal rays of $\mathcal{C}(0)$ are of the form $(1_j, -1_i)$, with $i \in N$ such that $|J_i| > 1$, $j \in J_i$ and $j \prec i$.

Here we use the notation 1_i for the vector of $\mathbb{R}^N$ having component i equals to 1 and 0 otherwise, and similarly for $(1_j, -1_i)$, etc.

Recall that $\mathcal{C}(v)$ will become bounded if there is no more extremal rays in $\mathcal{C}(0)$. Therefore, we must study how inequalities turned into equalities can "kill" extremal rays of $\mathcal{C}(0)$.

The following can be proved.

Lemma 1. *Consider J_i, $|J_i| > 1$, $j \prec i$. The extremal ray $(1_j, -1_i)$ is killed by equality $x(F) = 0$ if and only if $j \in F$ and $i \notin F$.*

Proof. ($\Leftarrow$) Suppose that $j \in F$ and $i \notin F$. Then, if $x \in \mathcal{C}(0)$ satisfies $x(F) = 0$, we have

$$x(F) = x_j + \sum_{k \in F \setminus j} x_k = 0. \tag{1}$$

Consider now $x' := x + \alpha(1_j, -1_i)$, $\alpha > 0$. Then x' does not satisfy equality $x'(F) = 0$ since

$$x'(F) = x_j + \alpha + \sum_{k \in F \setminus j} x_k = \alpha.$$

Therefore, $(1_j, -1_i)$ is no more a ray.

($\Rightarrow$) Let $x \in \mathcal{C}(0)$ and satisfy the additional equality $x(F) = 0$ for some $F \in \mathcal{F}$. Suppose that for some $\alpha > 0$, $x' := x + \alpha(1_j, -1_i)$ does not belong to $\mathcal{C}(v) \cap \{x(F) = 0\}$. It means that

$$\sum_{k \in F} x'_k = \sum_{k \in K} x_k + \alpha\delta_F(j) - \alpha\delta_F(i) \neq 0,$$

where $\delta_F(k) = 1$ if $k \in F$ and 0 otherwise. This implies $\delta_F(i) \neq \delta_F(j)$, therefore either i or j belongs to F, but not both. Because $j \prec i$ and that a set $F \in \mathcal{F}$ corresponds to a downset in $(N, \leq)$, it must be $j \in F$ and $i \notin F$. □

Lemma 2. *The minimum number of additional equalities needed to make $\mathcal{C}(v)$ bounded is $h(N)$.*

Proof. Let us assume that all rays are killed. By definition of the height, there exists a maximal chain in $(N, \leq)$ of length $h(N)$ going from a minimal element to a maximal element, say $i_0, i_1, \ldots, i_{h(N)}$. Then by Theorem 1, $(1_{i_0}, -1_{i_1})$, $(1_{i_1}, -1_{i_2})$, ..., $(1_{i_{h(N)-1}}, -1_{i_{h(N)}})$ are extremal rays. Because $(1_{i_0}, -1_{i_1})$ is killed, by Lemma 1 there must be an equality $x(K_1) = 0$ such that $i_0 \in K_1$ and $i_1 \notin K_1$. Moreover, since K_1 must be a downset, $i_2, \ldots, i_{h(N)}$ cannot belong to K_1. Similarly, there must exist an equality $x(K_2) = 0$ killing ray $(1_{i_1}, -1_{i_2})$ such that $i_1 \in K_2$ and $i_2, \ldots, i_{h(N)} \notin K_2$. Therefore, $K_1 \neq K_2$. Continuing this process we construct a sequence of distinct $h(N)$ subsets $K_1, K_2, \ldots, K_{h(N)}$, the last one killing ray $(1_{i_{h(N)-1}}, -1_{i_{h(N)}})$. Therefore, at least $h(N)$ equalities are needed. □

The next algorithm shows an optimal way to define equalities to kill all extremal rays. It is optimal in the sense that it uses only $h(N)$ equalities and each equality is the "smallest" possible (in the number of terms, or equivalently, in the size of F).

Algo 1

Step 0 Initialization. Set $M = N$.

Step 1 Remove all disconnected elements from M (i.e., those elements which are both minimal and maximal). If $M = \emptyset$, then STOP. Otherwise, go to Step 2.

Step 2 Build M_0 the set of all minimal elements of M, and set equality $x(\downarrow M_0) = 0$, where $\downarrow M_0$ is the downset generated by M_0 in $(N, \leq)$.

Step 3 Set $M \leftarrow M \setminus M_0$, and go to Step 1.

Theorem 2. Algo 1 kills all extremal rays and is optimal.

Proof. Steps 1 and 2 build subsets of the level sets of $(N, \leq)$, except the last $h(N)$th level, because in Step 2, all maximal elements of N are suppressed. Therefore, the algorithm necessarily finishes in exactly $h(N)$ iterations, and builds $h(N)$ equalities. By Lemma 2, this number is optimal.

Consider the first occurrence of Step 2, where M_0 is the set of minimal elements of N (minus those disconnected). Clearly, the equality $x(M_0) = 0$ kills all rays of the form $(1_j, -1_i)$, where j is a minimal element and i is a successor of j (i.e., $j \prec i$). Therefore, all such i's belong to the level 2. Taking a proper subset of M_0 will necessarily leave some rays of this form, and subsequent iterations will not kill them. This proves that in each step M_0 has a minimal size.

For each iteration, it is not necessary to keep elements i which have no successors (i.e., they are maximal), because there cannot exist rays of the form $(1_i, -1_k)$. Therefore those elements are suppressed in Step 1. All other elements are necessary since they have a successor and therefore generate a ray. This proves that in any iteration, M_0 has the minimal size, and so $\downarrow M_0$ too. □

We call the normal collection $\mathcal{N}$ found by Algo 1 the collection of *irredundant normal sets* or *irredundant (normal) collection*. We introduce another one, which we call the collection of *Weber normal sets* or the *Weber (normal) collection* (reasons for this will be clear after). Supposing $\mathcal{N} = \{N_1, \ldots, N_{h(N)}\}$ is the irredundant collection, the Weber collection is $\{N_1, N_1 \cup N_2, N_1 \cup N_2 \cup N_3, \ldots, N_1 \cup \cdots \cup N_{h(N)}\}$.

Lemma 3. The Weber collection is a normal collection which is a chain in $\mathcal{O}(N)$.

Proof. Lemma 1 shows that the collection is normal (only elements below those in the irredundant sets are added). The second assertion is obvious by construction. □

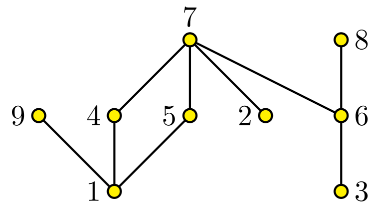
Recall that the normal collection introduced by Grabisch and Xie is $(L_1, L_1 \cup L_2, \ldots, L_1 \cup \cdots \cup L_{q-1})$, where $L_1, \ldots, L_q$ are the level sets of $(N, \leq)$. By construction, $N_1 \subseteq L_1$, $N_2 \subseteq L_1 \cup L_2$, etc., with proper inclusion in general. This shows that in general the three normal collections introduced so far differ.

When a normal collection forms a chain, we say that the collection is *nested*. Note that the Weber collection is the "smallest" nested collection, in the sense that no other nested collection can contain proper subsets of the Weber collection. Indeed, it is built from the irredundant normal collection by adding the minimum number of elements to make the collection a chain.

Interestingly, the normal collection of Grabisch and Xie is also nested, and it is the "largest" nested collection[1], in the sense that no other nested collection can contain supersets of this normal collection. Indeed, since a normal set is built from the union of all level sets up to a given height, adding a new element i means adding an element from a higher level. Then $(1_k, -1_i)$ for some $k \prec i$ is an extremal ray, which will not be killed if i is incorporated into the normal set. Consequently, any nested collection (with optimal number of normal sets) is comprised between the Weber collection and the Grabisch-Xie collection.

The following example illustrates that the three normal collections differ.

Example 1. Consider the following poset $(N, \leq)$ of 9 elements.



Level 1 is $\{1, 2, 3\}$, level 2 is $\{4, 5, 6, 9\}$ and level 3 is $\{7, 8\}$. Extremal rays are

$$(1_1, -1_9), (1_1, -1_4), (1_1, -1_5), (1_3, -1_6), (1_4, -1_7), (1_5, -1_7), (1_2, -1_7), (1_6, -1_7), (1_6, -1_8).$$

The two irredundant normal sets built by ALGO 1 are 123 and 13456, the two Weber normal sets are 123 and 123456, and the Grabisch-Xie normal sets are 123 and 1234569.

3.2 The Weber set

Let us denote by $\mathcal{C}$ the set of all maximal chains from $\emptyset$ to N in $\mathcal{F}$. Consider any maximal chain $C \in \mathcal{C}$ and its associated permutation σ on N, i.e.,

$$C = \{\emptyset, S_1, S_2, \ldots, S_n = N\},$$

with $S_i := \{\sigma(1), \ldots, \sigma(i)\}$, $i = 1, \ldots, n$. Considering a game v on $\mathcal{F}$, the *marginal vector* $x^C \in \mathbb{R}^N$ associated to C is the payoff vector defined by

$$x^C_{\sigma(i)} := v(S_i) - v(S_{i-1}), \quad i \in N.$$

The *Weber set* is the convex hull of all marginal vectors:

$$\mathcal{W}(v) := \mathrm{conv}(x^C \mid C \in \mathcal{C}).$$

In the classical case $\mathcal{F} = 2^N$, it is well known that for any game v it holds $\mathcal{C}(v) \subseteq \mathcal{W}(v)$, with equality if and only if v is convex. In our general case, this inclusion cannot hold any more since the core is unbounded in general. We propose a restricted version of the Weber set so that the classical results still hold.

Consider a nested normal collection (like the Weber collection or the Grabisch-Xie one) $\mathcal{N} = \{N_1, \ldots, N_{h(N)}\}$. A *restricted maximal chain* (w.r.t. $\mathcal{N}$) is a maximal chain

[1] Note that this collection is still optimal in number of normal sets. "Largest" applies here for the size of the normal sets.

from $\emptyset$ to N in $\mathcal{O}(N)$ containing $\mathcal{N}$. A *restricted marginal vector* is a (classical) marginal vector whose underlying maximal chain is restricted. The *(restricted) Weber set* $\mathcal{W}_{\mathcal{N}}(v)$ is the convex hull of all restricted marginal vectors w.r.t. $\mathcal{N}$. The (unrestricted) Weber set corresponds to the situation $\mathcal{N} = \emptyset$.

Lemma 4. For any restricted maximal chain C, its associated restricted marginal vector x^C coincides with v on C, i.e., $x^C(S) = v(S)$ for all $S \in C$.

(obvious from the definition)

We recall the following result (see Fujishige and Tomizawa [14, 13]).

Theorem 3. Let v be a game on $\mathcal{O}(N)$. Then $\mathcal{C}(v) = \mathcal{W}(v)$ if and only if v is convex.

The following theorems generalize results of [18] and provide more elegant proofs.

Theorem 4. Consider $\mathcal{N}$ a nested normal collection. Then for every game v on $\mathcal{O}(N)$, $\mathcal{C}_{\mathcal{N}}(v) \subseteq \mathcal{W}_{\mathcal{N}}(v)$.

Proof. We put $\mathcal{N} := \{N_1, \dots, N_q\}$. We prove the result by the separation theorem, proceeding as in [9]. Suppose there exists $x \in \mathcal{C}_{\mathcal{N}}(v) \setminus \mathcal{W}_{\mathcal{N}}(v)$. Then it exists $y \in \mathbb{R}^n$ such that $\langle w, y\rangle > \langle x, y\rangle$ for all $w \in \mathcal{W}_{\mathcal{N}}(v)$.

Let π be a permutation on N such that $y_{\pi(1)} \geq y_{\pi(2)} \geq \cdots \geq y_{\pi(n)}$. Let us build a permutation π' from π so that π' corresponds to a restricted maximal chain as follows:

> Order the elements of N_1 according to the π order; then order the elements of N_2 according to the π order and put them after ; etc. Lastly, put the remaining elements (in $N \setminus (N_1 \cup \cdots \cup N_q)$) according to the π order.

Note that $\pi' = \pi$ if π corresponds to a restricted maximal chain. With Example 1 and the Weber collection, taking $\pi = 1, 4, 5, 2, 9, 3, 6, 7, 8$ leads to $\pi' = 1, 2, 3, 4, 5, 6, 9, 7, 8$.

Denoting by $m^{\pi'}$ the marginal vector associated to π' we have

$$\begin{aligned}
\langle m^{\pi'}, y\rangle &= \sum_{i=1}^{n} y_{\pi'(i)}\big(v(\{\pi'(1), \dots, \pi'(i)\}) - v(\{\pi'(1), \dots, \pi'(i-1)\})\big) \\
&= y_{\pi'(n)}v(N) + \sum_{i=1}^{n-1}(y_{\pi'(i)} - y_{\pi'(i+1)})v(\{\pi'(1), \dots, \pi'(i)\}).
\end{aligned}$$

We claim that if $y_{\pi'(i)} - y_{\pi'(i+1)} < 0$ then $\{\pi'(1), \dots, \pi'(i)\}$ is a normal set. Indeed, by construction of π', the situation $y_{\pi'(i)} - y_{\pi'(i+1)} < 0$ can arise only if $\pi'(i) \in N_j$ for some j and $\pi'(i+1) \in N_{j+1}$. But then by construction again $N_j = \{\pi'(1), \dots, \pi'(i)\}$, which proves the claim.

Therefore since $x \in \mathcal{C}_{\mathcal{N}}(v)$ we have

$$\begin{aligned}
\langle m^{\pi'}, y\rangle &\leq y_{\pi'(n)}x(N) + \sum_{i=1}^{n-1}(y_{\pi'(i)} - y_{\pi'(i+1)})x(\{\pi'(1), \dots, \pi'(i)\}) \\
&= \sum_{i=1}^{n} y_{\pi'(i)}x(\{\pi'(1), \dots, \pi'(i)\}) - \sum_{i=2}^{n} y_{\pi'(i)}x(\{\pi'(1), \dots, \pi'(i-1)\}) \\
&= \sum_{i=1}^{n} y_{\pi'(i)}x_{\pi'(i)} = \langle y, x\rangle,
\end{aligned}$$

a contradiction with the assumption. ☐

Theorem 5. Consider $\mathcal{N}$ a nested normal collection. If v is convex on $\mathcal{O}(N)$, then $\mathcal{C}_{\mathcal{N}}(v) = \mathcal{W}_{\mathcal{N}}(v)$.

Proof. By Theorem 4, it suffices to show that any restricted marginal vector is a vertex of $\mathcal{C}_{\mathcal{N}}(v)$. We know already from Theorem 3 that it is a vertex of $\mathcal{C}(v)$. It remains to show that any marginal vector satisfies the normality conditions $x(N_i) = v(N_i)$, $i = 1, \ldots, q$, but this is established in Lemma 4. □

4 The general case

We suppose now that $\mathcal{F}$ is an arbitrary set system. We introduce $\widetilde{\mathcal{F}}$ the closure of $\mathcal{F}$ under union and intersection, i.e., the smallest set system closed under union and intersection containing $\mathcal{F}$. It is obtained by iteratively augmenting $\mathcal{F}$ with unions and intersections of pairs of subsets of the current set system (starting with $\mathcal{F}$), till there is no more change in the set system. As in Section 3, we assume that $\widetilde{\mathcal{F}}$ has height n (i.e., it has n join-irreducible elements).

Theorem 6. Consider an arbitrary set system $\mathcal{F}$, and assume that its closure $\widetilde{\mathcal{F}}$ has height n. Denote by $\mathcal{C}(0)$ and $\widetilde{\mathcal{C}}(0)$ the recession cones generated by $\mathcal{F}$ and $\widetilde{\mathcal{F}}$. Then $\mathcal{C}(0)$ and $\widetilde{\mathcal{C}}(0)$ have the same extremal rays (i.e., $\mathcal{C}(0) = \widetilde{\mathcal{C}}(0)$) if and only if all extremal rays of $\mathcal{C}(0)$ are of the form $(1_j, -1_i)$, for some $i, j \in N$.

Proof. The "only if" part is obvious from Theorem 1. Let us prove the "if" part. Suppose r is an extremal ray of $\mathcal{C}(0)$. By hypothesis, it has the form $(1_j, -1_i)$ for some $i, j \in N$. Also, by definition, it satisfies the system $r(S) \geq 0$ for all $S \in \mathcal{F}$, which gives $1_S(j) - 1_S(i) \geq 0$ for all $S \in \mathcal{F}$, which implies that there is no $S \in \mathcal{F}$ such that $S \ni i$ and $S \not\ni j$. Therefore it suffices to show that no such S exists in $\widetilde{\mathcal{F}}$. We show this by induction since $\widetilde{\mathcal{F}}$ is obtained iteratively from $\mathcal{F}$. We first prove that the union or intersection of two sets S_1, S_2 of $\mathcal{F}$ cannot at the same time contain i and not j. For intersection, if $S_1 \cap S_2 \ni i$, then S_1, S_2 too, so they cannot contain j, which implies $S_1 \cap S_2 \not\ni j$. Now, suppose that $S_1 \cup S_2$ does not contain j, which implies that neither S_1 nor S_2 contain j. If $i \in S_1 \cup S_2$, then i belongs at least to one of the sets S_1, S_2, which contradicts the hypothesis. Assume now that the hypothesis holds up to some step in the iteration process. Clearly, the same reasoning applies again, which proves that r is a ray of $\widetilde{\mathcal{C}}(0)$. Hence we have proved $\mathcal{C}(0) \subseteq \widetilde{\mathcal{C}}(0)$.

Conversely, suppose r is an extremal ray of $\widetilde{\mathcal{C}}(0)$, hence of the form $(1_j, -1_i)$ by Theorem 1. Then it satisfies the system $r(S) \geq 0$ for all $S \in \widetilde{\mathcal{F}}$, and $r(N) = 0$. Hence in particular it satisfies the system $r(S) \geq 0$ for all $S \in \mathcal{F}$ and $r(N) = 0$, and therefore r is a ray of $\mathcal{C}(0)$. Therefore $\widetilde{\mathcal{C}}(0) \subseteq \mathcal{C}(0)$. Hence, we have proved $\mathcal{C}(0) = \widetilde{\mathcal{C}}(0)$ and so extremal rays of $\mathcal{C}(v)$ and $\widetilde{\mathcal{C}}(v)$ are identical. □

Unfortunately, not all set systems $\mathcal{F}$, even if $\widetilde{\mathcal{F}}$ has height n, induce extremal rays of the form $(1_j, -1_i)$, as shown in the next example.

Example 2. Consider $N = \{1, 2, 3, 4\}$, the following set system $\mathcal{F}$ and its closure $\widetilde{\mathcal{F}}$. The extremal rays of $\mathcal{F}$ are $(1, -1, 1, -1)$, $(-1, 1, -1, 1)$ and $(0, 0, 1, -1)$, while the extremal

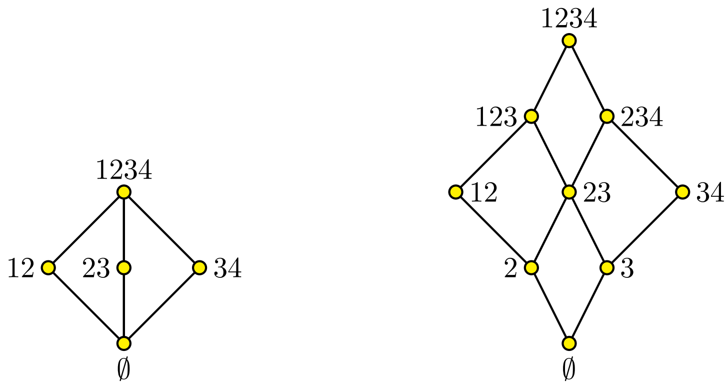


Figure 2: Set system $\mathcal{F}$ (left) and its closure under union and intersection $\widetilde{\mathcal{F}}$ (right)

rays of $\widetilde{\mathcal{F}}$ are $(-1, 1, 0, 0)$ and $(0, 0, 1, -1)$. Note that the first two rays of $\mathcal{F}$ in fact define a line, and that $\mathcal{F}$ is neither regular nor weakly union-closed.

Suppose now that $\mathcal{F}$ has rays of the form $(1_j, -1_i)$. How to kill them? Lemma 1 tells us how to kill rays of $\mathcal{F}$, by considering the equality $x(F) = 0$ with $j \in F$ and $i \notin F$. Therefore, the only thing we have to prove is that in any case, such a set F exists in $\mathcal{F}$.

Lemma 5. Let $\mathcal{F}$ be a set system such that all extremal rays of $\mathcal{C}(0)$ are of the form $(1_j, -1_i)$. Then for each extremal ray $(1_j, -1_i)$, there exists a set $F \in \mathcal{F}$ such that $j \in F$ and $i \notin F$.

Proof. We consider the ray $(1_j, -1_i)$. We know that in $\widetilde{\mathcal{F}}$ it exists F_0 such that $j \in F_0$ and $i \notin F_0$. Suppose that no such F exists in $\mathcal{F}$ and show that in this case F_0 cannot exist in $\widetilde{\mathcal{F}}$. We suppose therefore that in $\mathcal{F}$ all sets satisfy either $F \not\ni j$ or $F \ni i$ and we consider two sets F_1, F_2. Observe that we have four possible situations: 1) $F_1 \not\ni j$ and $F_2 \not\ni j$, 2) $F_1 \ni i, j$ and $F_2 \ni i, j$ 3) $F_1 \not\ni j$ and $F_2 \ni i, j$, and 4) $F_1 \ni i, j$ and $F_2 \not\ni j$. In all four situations, we cannot have both $F_1 \cup F_2 \ni j$ and $F_1 \cup F_2 \not\ni i$, and the same is true for $F_1 \cap F_2$. Therefore, after one iteration, the set system has the same property than $\mathcal{F}$, and so by successive iterations, F_0 cannot be built. $\square$

The above lemma tells us that it is possible to kill rays for such set systems by turning at most r inequalities to equalities, if r is the number of rays. Is it possible to give a better answer by using results from Section 3.1 on $\widetilde{\mathcal{F}}$? Unfortunately, it does not seem possible to give a general answer here, even for regular set systems. This is because the irredundant normal sets found by ALGO 1 or the Weber normal collection of $\widetilde{\mathcal{F}}$ need not belong to $\mathcal{F}$, as the following simple example shows.

Example 3. Consider $N = \{1, 2, 3, 4\}$, the following set system $\mathcal{F}$ (which is regular) and its closure $\widetilde{\mathcal{F}}$. The unique ray of $\mathcal{C}(0)$ is (0,0,1,-1). Application of ALGO 1 on $\widetilde{\mathcal{F}}$ gives as normal set 3 (the Weber normal set is therefore the same). However, 3 does not belong to $\mathcal{F}$. Either 13 or 23 can be taken instead. Note that the Grabisch-Xie normal set is 123, which does not belong either to $\mathcal{F}$.

Hence, the only thing which can be done is to build $\widetilde{\mathcal{F}}$, apply ALGO 1 or compute the Weber normal collection. If some normal sets do not belong to $\mathcal{F}$, take the smallest

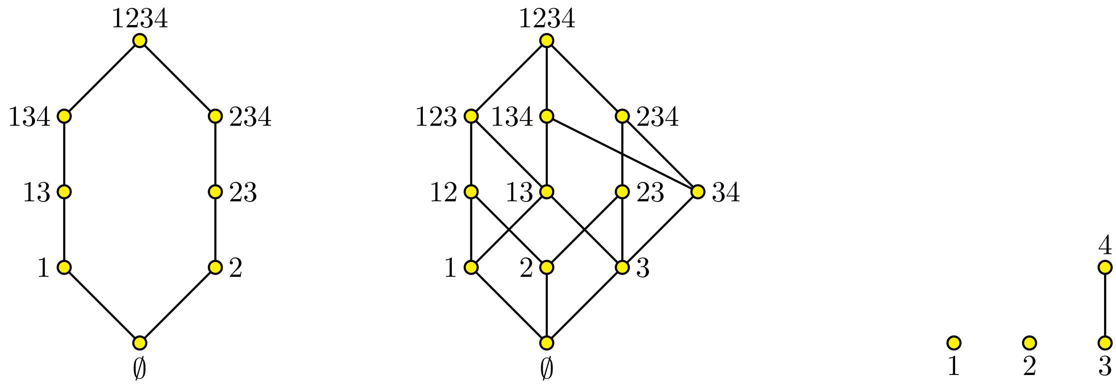


Figure 3: Set system $\mathcal{F}$ (left), its closure under union and intersection $\widetilde{\mathcal{F}}$ (center), and the generating poset $(N, \leq)$ (right)

ones of $\mathcal{F}$ containing them and obeying Lemma 1. It is not guaranted however that we do not need more normal sets than for $\widetilde{\mathcal{F}}$ (but we do not have an example for this).

In the rest of the paper, we study two particular types of sets systems, namely regular set systems and weakly union-closed set systems, which both generalize systems closed under union and intersection, and where the above results can be applied.

4.1 The case of regular set systems

Recall that any maximal chain induces a total order (permutation) on N, and therefore giving a regular set system $\mathcal{F}$ is equivalent to giving a set of (permitted) total orders on N.

Theorem 7. Suppose $\mathcal{F}$ is a regular set system. Then all extremal rays of $\mathcal{C}(0)$ have the form $(1_l, -1_m)$ for some $l, m \in N$.

Proof. Let $\mathfrak{C}$ be the set of all maximal chains from $\emptyset$ to N in $\mathcal{F}$, and consider a particular chain, say $\emptyset, \{i\}, \{i,j\}, \{i,j,k\}, \ldots, N$, inducing the total order $i, j, k, \ldots,$ on N, and let us construct an extremal ray r.

Suppose $r_i > 0$, hence w.l.o.g. we can set $r_i = 1$. By the condition $r(N) = 0$, there must be at least one $\ell \in N \setminus i$ such that $r_\ell < 0$. Select ℓ such that ℓ is ranked after i in every maximal chain in $\mathfrak{C}$. Observe that $(1_i, -1_\ell)$ is a solution of the system $r(S) \geq 0$ for all $S \in \mathcal{F}$ and $r(N) = 0$ (i.e., it is a ray of $\mathcal{C}(0)$) if and only if ℓ has the above property, because any $S \ni \ell$ contains also i. If no such ℓ exists, then set $r_i = 0$, which gives a new system of inequalities where r_i has disappeared, and consider the next element j and do the same (note that if exhausting all elements $i, j, k, \ldots$ without finding ℓ, is equivalent to the fact that there is no ray, a situation which happens for example if all orders exist, i.e., $\mathcal{F} = 2^N$). Suppose now that there exist several ℓ ranked after i in every maximal chain, say $\ell_1, \ldots, \ell_q$. Then for every $\alpha_1, \ldots, \alpha_q \geq 0$ such that $\sum_{p=1}^q \alpha_p = 1$, the vector $(1_i, -\alpha_1 1_{\ell_1}, \ldots, -\alpha_q 1_{\ell_q})$ is a ray. But each $(1_i, -1_{\ell_p})$, $p = 1, \ldots, q$ is also a ray, and $(1_i, -\alpha_1 1_{\ell_1}, \ldots, -\alpha_q 1_{\ell_q})$ can be expressed as a convex combination of these rays, proving that it is not extremal. Therefore extremal rays are necessarily of the form $(1_i, -1_\ell)$. In addition, if ℓ_2 is ranked after ℓ_1 in every order, then $(1_{\ell_1}, -1_{\ell_2})$ is a ray, therefore $(1_i, -1_{\ell_2})$

is not extremal since it can be obtained as $(1_i, -1_{\ell_1}) + (1_{\ell_1}, -1_{\ell_2})$ (and similarly for the others). $\square$

By Theorem 6, we deduce immediately:

Corollary 1. If $\mathcal{F}$ is a regular set system, then $\mathcal{C}(0) = \widetilde{\mathcal{C}}(0)$.

We can also deduce Theorem 1 from the above, and therefore derive an alternative proof of it:

Corollary 2. If $\mathcal{F}$ is regular and union and intersection closed, then the extremal rays are $(1_j, -1_i)$ with $i \in N$ such that $|J_i| > 1$ and $j \in J_i$, $j \prec i$.

Proof. Under the hypothesis, $\mathcal{F}$ is generated by a poset $(N, \leq)$, and the set of total orders generated by the maximal chains are those orders compatible with the partial order $\leq$ on N. Then it is easy to see from the proof of Theorem 7 that we obtain the desired extremal rays. $\square$

The proof of Theorem 7 being constructive, we can propose the following simple algorithm to produce all extremal rays of a regular set system.

ALGO 2

Step 0 Initialization. Select a maximal chain C in $\mathfrak{C}$, and denote for simplicity by $1, 2, \ldots, n$ the order induced by C. Put $L = \emptyset$.

For $i = 1$ to $n - 1$ **do**:

For $j = i + 1$ to n **do**:

If j is ranked after i in every chain in $\mathfrak{C}$, **then**

- Put $(1_i, -1_j)$ in L
  % this is a candidate for being an extremal ray
- **For** $k < i$, check if $(1_k, -1_i)$ and $(1_k, -1_j)$ both exist in L. **If** yes, remove $(1_k, -1_j)$ from L
  % it can be obtained as the sum of $(1_k, -1_i)$ and $(1_i, -1_j)$

Final step: output list L of extremal rays.

Example 4. Let us apply ALGO 2 on the regular set system of Fig. 4 (left). The four orders induced by the maximal chains are:

$$1 - 4 - 2 - 3 - 5$$
$$2 - 4 - 1 - 3 - 5$$
$$2 - 4 - 3 - 5 - 1$$
$$2 - 4 - 3 - 1 - 5$$

Let us take the first order for running the algorithm. Taking $i = 1$, we see that no j can be found. Therefore, we take $i = 4$, then $j = 3$ and 5 are possible, so we put in L the rays $(0, 0, -1, 1, 0)$ and $(0, 0, 0, 1, -1)$. Let us take now $i = 2$, then $j = 3$ and 5 are possible, so we add in L the two rays $(0, 1, -1, 0, 0)$ and $(0, 1, 0, 0 - 1)$. Next, we take $i = 3$ and see that $j = 5$ is possible, therefore we put $(0, 0, 1, 0, -1)$ in L. However, we have to remove $(0, 0, 0, 1, 0, -1)$ and $(0, 1, 0, 0, -1)$ from L. The extremal rays are therefore $(0, 0, -1, 1, 0)$, $(0, 1, -1, 0, 0)$ and $(0, 0, 1, 0, -1)$. This result is confirmed by the PORTA software.

We end this section by addressing the definition of the Weber set. Since $\mathcal{F}$ is regular, marginal vectors can be defined as usual and therefore it makes sense to speak of the Weber set. Suppose we have found a normal nested collection of sets $\mathcal{N}$, then the restricted Weber set $\mathcal{W}_{\mathcal{N}}(v)$ for v defined on $\mathcal{F}$ can be defined as before. The question is then to compare $\mathcal{W}_{\mathcal{N}}(v)$ with $\mathcal{C}_{\mathcal{N}}(v)$ and also $\widetilde{\mathcal{W}}_{\mathcal{N}'}(v)$, the restricted Weber set on $\widetilde{\mathcal{F}}$, with $\mathcal{N}'$ the Weber normal collection of $\widetilde{\mathcal{F}}$. Little can be said in general if one does not have $\mathcal{N}' = \mathcal{N}$. Suppose then that this is the case. Because of regularity, any restricted maximal chain in $\mathcal{F}$ is a restricted maximal chain in $\widetilde{\mathcal{F}}$, so that we have $\mathcal{W}_{\mathcal{N}}(v) \subseteq \widetilde{\mathcal{W}}_{\mathcal{N}}(v)$. Recall also that $\mathcal{C}_{\mathcal{N}}(v) \supseteq \widetilde{\mathcal{C}}_{\mathcal{N}}(v)$, hence the question whether $\mathcal{C}_{\mathcal{N}}(v) \subseteq \mathcal{W}_{\mathcal{N}}(v)$ remains. An examination of the proof of Theorem 4 reveals that the technique of the proof cannot extend to this case. Indeed, the following example shows that this is not true in general.

Example 5. Consider $N = \{1, 2, 3, 4, 5\}$, the following regular set system $\mathcal{F}$ and its closure $\widetilde{\mathcal{F}}$. Algo 1 applied on $\widetilde{\mathcal{F}}$ gives 24 and 234 as normal sets, which is also the

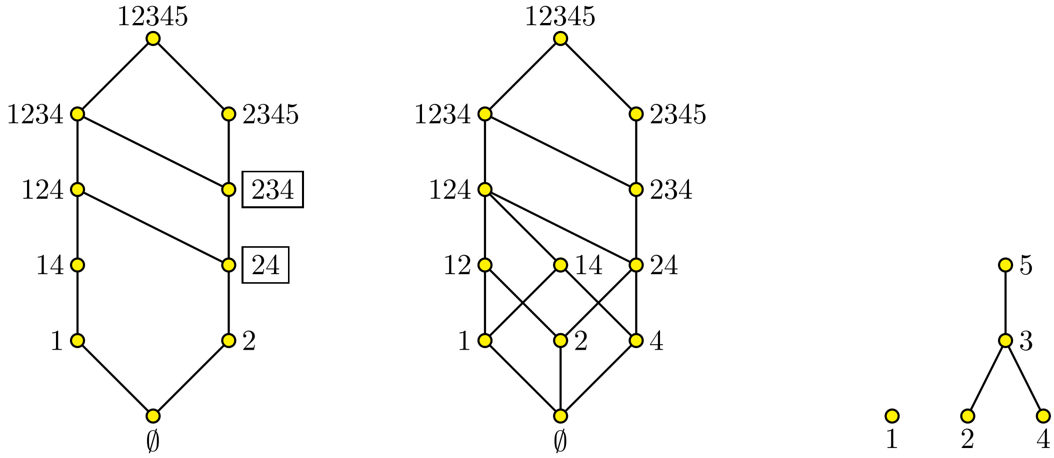


Figure 4: Set system $\mathcal{F}$ (left), its closure under union and intersection $\widetilde{\mathcal{F}}$ (center), and the generating poset $(N, \leq)$ (right)

Weber collection. These sets belong to $\mathcal{F}$, therefore the restricted Weber set can be defined with the Weber collection. There are only two restricted maximal chains on $\mathcal{F}$, namely $\emptyset, 2, 24, 234, 2345, N$ and $\emptyset, 2, 24, 234, 1234, N$, inducing the two vertices of $\mathcal{W}_{\mathcal{N}}(v)$:

$$w_1 = (v(N) - v(2345), v(2), v(234) - v(24), v(24) - v(2), v(2345) - v(234))$$
$$w_2 = (v(1234) - v(234), v(2), v(234) - v(24), v(24) - v(2), v(N) - v(1234)).$$

The restricted core is defined by the system:

$$\begin{aligned} x_1 &\geq v(1) \\ x_2 &\geq v(2) \\ x_1 + x_4 &\geq v(14) \\ x_2 + x_4 &= v(24) \\ x_1 + x_2 + x_4 &\geq v(124) \\ x_2 + x_3 + x_4 &= v(234) \\ x_1 + x_2 + x_3 + x_4 &\geq v(1234) \\ x_2 + x_3 + x_4 + x_5 &\geq v(2345) \\ x_1 + x_2 + x_3 + x_4 + x_5 &= v(N) \end{aligned}$$

Let us take the game defined by $v(N) = 3$, $v(1234) = v(2345) = 2$, $v(234) = 1$, $v(124) = 2$, $v(24) = v(14) = 1$, $v(2) = v(1) = 0$. Then the two vertices of the Weber set are $(1, 0, 0, 1, 1)$ and $(1, 0, 0, 1, 1)$, which makes the Weber set a singleton. However, the vector $(1, 1, 0, 0, 1)$ is an element of the restricted core, which forbids the core to be included into the Weber set.

4.2 The case of weakly union-closed systems

The situation here is less simple than with regular set systems. The following theorem gives a sufficient condition for the equality of $\mathcal{C}(0)$ and $\widetilde{\mathcal{C}}(0)$.

Theorem 8. Assume that $\mathcal{F}$ is a weakly union-closed system, and denote by $\widetilde{\mathcal{F}}$ its closure under union and intersection. Then the extremal rays of $\mathcal{C}(0)$ and $\widetilde{\mathcal{C}}(0)$ are the same if for any $S \in \widetilde{\mathcal{F}} \setminus \mathcal{F}$, it is either a union of disjoint sets of $\mathcal{F}$, or there exist $S_1, S_2 \in \mathcal{F}$ such that $S = S_1 \cap S_2$, and there exists a covering in $\mathcal{F}$ of $N \setminus (S_1 \cup S_2)$.

By definition of weakly union-closed systems, note that the covering will be in fact a partition.

Proof. We consider the set of inequalities of $\mathcal{C}(0)$, i.e., $x(S) \geq 0$ for all $S \in \mathcal{F}$ and $x(N) = 0$. We will prove that any additional inequality $x(F) \geq 0$ with $F \in \widetilde{\mathcal{F}} \setminus \mathcal{F}$ is redundant. By the Farkas lemma, we know that this amounts to prove that $x(F) \geq 0$ can be obtained by a positive linear combination of the inequalities $x(S) \geq 0$, $S \in \mathcal{F}$ and $x(N) = 0$.

We consider $S \in \widetilde{\mathcal{F}} \setminus \mathcal{F}$. Assume first that S is a disjoint union of sets in $\mathcal{F}$, say $S = S_1 \cup \cdots \cup S_k$. Then obviously $x(S) \geq 0$ is implied by equalities $x(S_i) \geq 0$, $i = 1, \ldots, q$, since it can be obtained as their sum. Suppose on the contrary that S is not a disjoint union of sets in $\mathcal{F}$. By hypothesis, there exists $S_1, S_2 \in \mathcal{F}$ such that $S_1 \cap S_2 = S$ and there exists a partition $\{T_1, \ldots, T_k\}$ of $N \setminus (S_1 \cup S_2)$. Let us write the following system of inequalities:

$$\begin{aligned}
x(S_1) &\geq 0 & & (a_1)\\
x(S_2) &\geq 0 & & (a_2)\\
x(T_1) &\geq 0 & & (b_1)\\
&\vdots & & \vdots\\
x(T_k) &\geq 0 & & (b_k)\\
-x(N) &\geq 0 & & (c),
\end{aligned}$$

the last one coming from $x(N) = 0$. Then the inequality $x(S) \geq 0$ is obtained by $(a_1) + (a_2) + (b_1) + \cdots (b_k) + (c)$, which proves that $x(S) \geq 0$ is redundant. $\square$

The next example illustrates the case where this condition is not satisfied.

Example 6. Take $N = \{1, 2, 3, 4\}$ and the following weakly union-closed set system $\mathcal{F}$ and its closure $\widetilde{\mathcal{F}}$. The required condition fails: take $S = 2$, then it can obtained only by

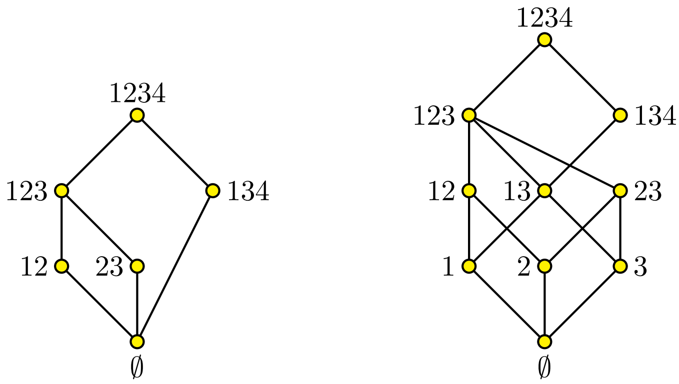


Figure 5: Set system $\mathcal{F}$ (left) and its closure under union and intersection $\widetilde{\mathcal{F}}$ (right)

the intersection of 12 and 23. But $N \setminus 123 = 4$ is not a subset of $\mathcal{F}$. The extremal rays of $\mathcal{C}(0)$ are $(0, 0, 1, -1)$, $(1, 0, 0, -1)$ and $(1, -1, 1, -1)$, but $\widetilde{\mathcal{C}}(0)$ has only the two first rays as extremal rays.

References

[1] E. Algaba, J. M. Bilbao, P. Borm, and J. J. López. The Myerson value for union stable structures. *Math. Meth. Oper. Res.*, 54:359–371, 2001.

[2] E. Algaba, J. M. Bilbao, R. van den Brink, and A. Jiménez-Losada. Cooperative games on antimatroids. *Discrete Mathematics*, 282:1–15, 2004.

[3] R. Amer, F. Carreras, and A. Magaña. Extension of values to games with multiple alternatives. *Annals of Operations Research*, 84:63–78, 1998.

[4] J. M. Bilbao, J. R. Fernández, N. Jiménez, and J. J. López. The Shapley value for bi-cooperative games. *Annals of Operations Research*, 158, 2008.

[5] J. M. Bilbao, E. Lebrón, and N. Jiménez. Probabilistic values on convex geometries. *Annals of Operations Research*, 84:79–95, 1998.

[6] G. Birkhoff. *Lattice Theory*. American Mathematical Society, 3d edition, 1967.

[7] E. M. Bolger. A consistent value for games with n players and r alternatives. *Int. J. of Game Theory*, 29:93–99, 2000.

[8] B. A. Davey and H. A. Priestley. *Introduction to Lattices and Orders*. Cambridge University Press, 1990.

[9] J.J.M. Derks. A short proof of the inclusion of the core in the Weber set. *Int. J. of Game Theory*, 21:149–150, 1992.

[10] J.J.M. Derks and H. Reijnierse. On the core of a collection of coalitions. *Int. J. of Game Theory*, 27:451–459, 1998.

[11] U. Faigle, M. Grabisch, and M. Heyne. Monge extensions of cooperation and communication structures. *European Journal of Operational Research*, to appear. 10.1016/j.ejor.2010.01.043.

[12] U. Faigle and W. Kern. The Shapley value for cooperative games under precedence constraints. *Int. J. of Game Theory*, 21:249–266, 1992.

[13] S. Fujishige. *Submodular functions and optimization*, volume 58 of *Annals of Discrete Mathematics*. Elsevier, Amsterdam, 2nd edition, 2005.

[14] S. Fujishige and N. Tomizawa. A note on submodular functions on distributive lattices. *J. of the Operations Research Society of Japan*, 26:309–318, 1983.

[15] M. Grabisch. The core of games on ordered structures and graphs. *4OR*, 7:207–238, 2009. DOI: 10.1007/s10288-009-0109-9.

[16] M. Grabisch and F. Lange. Games on lattices, multichoice games and the Shapley value: a new approach. *Mathematical Methods of Operations Research*, 65:153–167, 2007.

[17] M. Grabisch and L. J. Xie. A new approach to the core and Weber set of multichoice games. *Mathematical Methods of Operations Research*, 66:491–512, 2007. doi 10.1007/s00186-007-0159-8.

[18] M. Grabisch and L. J. Xie. The core of games on distributive lattices: how to share benefits in a hierarchy. *Technical Report, Centre d'Economie de la Sorbonne, 2008.77*, 2008.

[19] A. Honda and M. Grabisch. Entropy of capacities on lattices and set systems. *Information Sciences*, 176:3472–3489, 2006.

[20] A. Honda and M. Grabisch. An axiomatization of entropy of capacities on set systems. *Eur. J. of Operational Research*, 190:526–538, 2008.

[21] C. R. Hsiao and T. E. S. Raghavan. Shapley value for multichoice cooperative games, I. *Games and Economic Behavior*, 5:240–256, 1993.

[22] T. Ichiishi. Super-modularity: applications to convex games and to the greedy algorithm for LP. *J. Econom. Theory*, 25:283–286, 1981.

[23] Ch. Labreuche and M. Grabisch. A value for bi-cooperative games. *Int. J. of Game Theory*, 37:409–438, 2008. 10.1007/s00182-008-0126-5c.

[24] F. Lange and M. Grabisch. Values on regular games under Kirchhoff's laws. *Mathematical Social Sciences*, 58:322–340, 2009. DOI: 10.1016/j.mathsocsci.2009.07.003.

[25] M. Núñez and C. Rafels. On extreme points of the core and reduced games. *Annals of Operations Research*, 84:121–133, 1998.

[26] H. Peters and H. Zank. The egalitarian value for multichoice games. *Annals of Operations Research*, 137:399–409, 2005.

[27] L. S. Shapley. Core of convex games. *Int. J. Game Theory*, 1:11–26, 1971.

[28] J. Suijs, P. Borm, H. Hamers, M. Quant, and M. Koster. Communication and cooperation in public network situations. *Annals of Operations Research*, 137:117140, 2005.

[29] N. Tomizawa. Theory of hyperspace (XVI)—on the structure of hedrons. Papers of the Technical Group on Circuits and Systems CAS82-172, Inst. of Electronics and Communications Engineers of Japan, 1983. In Japanese.

[30] R. van den Brink, G. van der Laan, and V. Vasil'ev. Component efficient solutions in line-graph games with applications. *Economic Theory*, 33:349–364, 2007.